

General Wigner Rotations in D Dimensions

Fa-Min Chen

Department of Physics, Beijing Jiaotong University, Beijing 100044, China

ABSTRACT: We construct general Wigner rotations for both massive and massless particles in arbitrary D dimensions. We work out the explicit expressions for Wigner rotations for a general Lorentz transformation. We study the relation between the electromagnetic gauge invariance and the non-uniqueness of Wigner rotation.

Contents

1	Introduction and summary	1
2	Wigner Rotations for Massive Particles	2
2.1	D Dimensions	2
2.2	4 Dimensions	8
2.3	Summary of This Section	12
3	Wigner Rotations for Massless Particles	12
3.1	D Dimensions	12
3.2	4 Dimensions, and Applications to Gauge Theory	17
3.3	Summary of This Section	21
4	Acknowledgement	21
A	Conventions and Useful Identities	21

1 Introduction and summary

In quantum field theory, one-particle states are classified according to the representations of little groups of the Lorentz group [1]. For a systematic introduction of little groups of $4D$ Lorentz group, see Ref. [2].

For a chosen “standard” D -momentum k^ν ¹, the little group is defined as $W^\mu_\nu k^\nu = k^\mu$. ($\mu, \nu = 0, 1, \dots, D-1$.) For a given Lorentz transformation Λ and a given momentum p^μ , the little group can be constructed as [2]

$$W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) \quad (1.1)$$

Here $L(p)$ is a standard Lorentz transformation, bringing k^μ to p^μ , i.e. $p^\mu = L^\mu_\nu(p) k^\nu$.

In this paper, we work out the explicit expressions of little group elements (1.1) for both massive and massless particles in an arbitrary D dimensional spacetime.

Our main idea is to use spinor algebra to construct the little groups or Wigner rotations. Generally speaking, the spinor algebra in D dimensions is slightly easier than the tensor algebra. Nevertheless, the spinors can still furnish faithful representations of the little groups; So they can be used to work out (1.1). The technical details will be introduced in the next section.

¹For a particle of unit mass, $k^\mu = (0, 0, \dots, 0, 1)$; For a massless particle, $k^\mu = (0, \dots, 0, 1, 1)$, with “1” standing for unit energy.

For the massive particle case, we use two distinct methods to derive the the Wigner rotation in D dimensions; In $4D$, we provide a third way to work out the explicit expression for the Wigner rotation.

The spinor representation of little group for massless particle is particularly interesting. For instance, in $4D$, the little group is $ISO(2)$, with the rotation generator J^3 and two translation generators T^1 and T^2 . If the state is a superposition of the eigenvectors of T^1 and T^2 , and if the eigenvalues of T^1 and T^2 are not zero, the helicity σ of a massless fermionic particle would have a continuous value without taking account of the topology of the Lorentz group. However, in the spinor realization of $ISO(2)$, the eigenvalues of $A^1 \equiv T_s^1$ and $A^2 \equiv T_s^2$ are zero automatically. (Here “s” stands for spinor representation.) So that a continuous value of the helicity σ of a massless fermionic particle can be avoided, without even considering the topology of the Lorentz group [2].

It is obvious for a given Lorentz transformation, the Wigner rotation cannot be uniquely defined. For a fixed “standard” D -momentum k^μ , one may choose two different standard Lorentz transformations $L(p)$ and $\tilde{L}(p)$, in the sense that $L(p)^\mu{}_\nu k^\nu = \tilde{L}(p)^\mu{}_\nu k^\nu = p^\mu$ but $L(p) \neq \tilde{L}(p)$. The resulting two Wigner rotations satisfy

$$\widetilde{W}(\Lambda, p) = S(\Lambda p) W(\Lambda, p) S^{-1}(p) \quad (1.2)$$

where $S(p) \equiv \tilde{L}^{-1}(p)L(p)$. The above equation may be useful in studying gauge fields: Here $S(p)$ may have a connection with the gauge transformation of $U(1)$ gauge field in D dimensions. As an example, we discussed the relation between the ($4D$) electromagnetic gauge invariance and the non-uniqueness of Wigner rotation.

The results of this paper may be useful in studying theories in the higher dimensions, such as superstring theory or M-theory.

In the following two sections we construct the Wigner rotations for massive and massless particles, respectively.

2 Wigner Rotations for Massive Particles

2.1 D Dimensions

For a particle of mass M in D dimensions, we choose the standard vector as $k^\mu = (0, 0, \dots, 0, M)$. The spinor representation of the “standard boost” can be constructed as follows

$$L_s(\eta) = e^{\eta_i \Sigma^{i0}} \quad (2.1)$$

Here $\Sigma^{i0} = \frac{1}{4}[\gamma^0, \gamma^i]$ is the set of boost generators (our conventions are summarized in Appendix A), and η^i the set of rapidities ; The subscript “s” stands for spinor representation. The relation between $L_s(\eta)$ and $L(\eta)$ ² is the standard one:

$$L_s(\eta) \gamma^\mu L_s^{-1}(\eta) = L_\nu{}^\mu(\eta) \gamma^\nu. \quad (2.2)$$

²In this paper, Lorentz transformations without the subscript “s”, such as $L(p)$, Λ , R , and $W(\Lambda, p)$ are in the vector representation.

Using $(2\Sigma^{i0})^2 = 1$ (no sum), one can convert (2.1) into the form

$$L_s(\eta) = \cosh(\eta/2) + \sinh(\eta/2)\hat{\eta}^i(2\Sigma^{i0}), \quad (2.3)$$

where $\hat{\eta}^i \equiv \eta^i/\eta$ and $\eta \equiv |\vec{\eta}| = \sqrt{(\eta^i)^2}$. Substituting (2.3) into (2.2), we find that

$$\begin{aligned} L_i^j(\eta) &= \delta^{ij} + (\cosh \eta - 1)\hat{\eta}^i\hat{\eta}^j \\ L_0^i(\eta) &= L_i^0(\eta) = -\hat{\eta}^i \sinh \eta \\ L_0^0(\eta) &= \cosh \eta \end{aligned} \quad (2.4)$$

Substituting

$$\hat{\eta}^i = \hat{p}^i, \quad \sinh \eta = |\vec{p}|/M \quad (2.5)$$

into (2.4),

$$\begin{aligned} L_i^j(p) &= \delta^{ij} + (\gamma - 1)\hat{p}^i\hat{p}^j, \\ L_0^i(p) &= L_i^0(p) = -\hat{p}^i \sqrt{\gamma^2 - 1}, \\ L_0^0(p) &= \gamma, \end{aligned} \quad (2.6)$$

where $\gamma \equiv \sqrt{|\vec{p}|^2/M^2 + 1} = p^0/M$. We see that $L(\eta)$ or $L(p)$ does carry the D -momentum from k^μ to p^μ . Since now, we do not distinguish $L(\eta)$ and $L(p)$. It can be seen that if $D = 4$, the standard boost (2.4) is same as the one in Ref. [2].

For a given general Lorentz transformation Λ , we denote its spinor counterpart as Λ_s ,³ They satisfy the equation

$$\Lambda_s \gamma^\mu \Lambda_s^{-1} = \Lambda_\nu{}^\mu \gamma^\nu \quad (2.7)$$

Then the Wigner rotation in the spinor space reads

$$W_s(\Lambda, \eta) = L_s^{-1}(\eta_\Lambda) \Lambda_s L_s(\eta). \quad (2.8)$$

Here η_Λ must be defined such that $L(\eta_\Lambda)$ transform p^μ into $(\Lambda p)^\mu$, i.e.

$$\hat{\eta}_\Lambda^i = (\widehat{\Lambda p})^i, \quad \sqrt{((\Lambda p)^i)^2} = M \sinh(\eta_\Lambda). \quad (2.9)$$

This can be fulfilled by requiring that

$$\Lambda_s L_s(\eta) \gamma^0 L_s^{-1}(\eta) \Lambda_s^{-1} = L_s(\eta_\Lambda) \gamma^0 L_s^{-1}(\eta_\Lambda) \quad (2.10)$$

On one hand,

$$\Lambda_s L_s(\eta) \gamma^0 L_s^{-1}(\eta) \Lambda_s^{-1} = \Lambda_\nu{}^\mu L_\mu{}^0(\eta) \gamma^\nu. \quad (2.11)$$

On the other hand, in analogy to (2.3), we have

$$L_s(\eta_\Lambda) = \cosh(\eta_\Lambda/2) + \sinh(\eta_\Lambda/2)\hat{\eta}_\Lambda^i(2\Sigma^{i0}). \quad (2.12)$$

³If $D \leq 4$, it is relatively easy to work Λ_s for a given general Λ . (See Section 2.2.)

So

$$\begin{aligned} L_s(\eta_\Lambda)\gamma^0 L_s^{-1}(\eta_\Lambda) &= L_\nu^0(\eta_\Lambda)\gamma^\nu = \left(\cosh(\eta_\Lambda) + \sinh(\eta_\Lambda)\hat{\eta}_\Lambda^i(2\Sigma^{i0}) \right) \gamma^0 \\ &= \cosh(\eta_\Lambda)\gamma^0 - \sinh(\eta_\Lambda)\hat{\eta}_\Lambda^i\gamma^i. \end{aligned} \quad (2.13)$$

Comparing (2.11) and (2.13) gives

$$\begin{aligned} \cosh(\eta_\Lambda) &= (\Lambda L)_0^0 = \Lambda_0^0 \cosh(\eta) - \Lambda_0^i \hat{\eta}_i \sinh(\eta), \\ \hat{\eta}_\Lambda^j \sinh(\eta_\Lambda) &= -(\Lambda L)_j^0 = \Lambda_j^i \hat{\eta}_i \sinh(\eta) - \Lambda_j^0 \cosh(\eta). \end{aligned} \quad (2.14)$$

where we have used (2.4), and for readability, we have written $\Lambda_\mu{}^\rho L_\rho{}^\nu$ as $(\Lambda L)_\mu{}^\nu$.

On one hand, we have

$$L_s^{-1}(\eta_\Lambda) = \cosh(\eta_\Lambda/2) - \sinh(\eta_\Lambda/2)\hat{\eta}_\Lambda^i(2\Sigma^{i0}) \quad (2.15)$$

On the other hand, one can recast $L_s^{-1}(\eta_\Lambda)$ into the following form:

$$L_s^{-1}(\eta_\Lambda) = \frac{(\Lambda_s^\dagger)^{-1} L_s^{-2}(\eta) \Lambda_s^\dagger + 1}{2 \cosh(\eta_\Lambda/2)}. \quad (2.16)$$

To see this, let us first evaluate $\Lambda_s L_s^2 \Lambda_s^\dagger$. Using (2.3), (2.7), and $\Lambda_s^{-1} = \gamma^0 \Lambda_s^\dagger (\gamma^0)^{-1}$, we find that

$$\begin{aligned} \Lambda_s L_s^2 \Lambda_s^\dagger &= \Lambda_s [\cosh(\eta) + \sinh(\eta)\hat{\eta}^i(2\Sigma^{i0})] \Lambda_s^\dagger \\ &= [\Lambda_0^0 \cosh(\eta) - \Lambda_0^i \hat{\eta}^i \sinh(\eta)] + [\Lambda_j^i \hat{\eta}^i \sinh(\eta) - \Lambda_j^0 \cosh(\eta)] (2\Sigma^{j0}) \end{aligned} \quad (2.17)$$

So

$$\begin{aligned} &(\Lambda_s^\dagger)^{-1} L_s^{-2}(\eta) \Lambda_s^\dagger \\ &= \left(\Lambda_s L_s^2 \Lambda_s^\dagger \right)^{-1} = \gamma^0 \left(\Lambda_s L_s^2 \Lambda_s^\dagger \right)^\dagger (\gamma^0)^{-1} \\ &= [\Lambda_0^0 \cosh(\eta) - \Lambda_0^i \hat{\eta}^i \sinh(\eta)] - [\Lambda_j^i \hat{\eta}^i \sinh(\eta) - \Lambda_j^0 \cosh(\eta)] (2\Sigma^{j0}) \end{aligned} \quad (2.18)$$

Plugging the above equation into (2.16), and using (2.14), we find that (2.16) is exactly the same as (2.15).

Plugging (2.15) into (2.8) gives the spinor representation of the general Wigner rotation for massive particles in D dimension:

$$W_s(\Lambda, \eta) = \frac{(\Lambda_s^\dagger)^{-1} L_s^{-1}(\eta) + \Lambda_s L_s(\eta)}{2 \cosh(\eta_\Lambda/2)} = \frac{\gamma^0 \Lambda_s L_s(\eta) (\gamma^0)^{-1} + \Lambda_s L_s(\eta)}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}}, \quad (2.19)$$

where we have written the denominator as

$$2 \cosh(\eta_\Lambda/2) = \sqrt{2(\cosh(\eta_\Lambda) + 1)} = \sqrt{2(1 + [\Lambda L(p)]_0^0)}. \quad (2.20)$$

It is easy to check that

$$W_s^\dagger(\Lambda, \eta) = W_s^{-1}(\Lambda, \eta) \quad (2.21)$$

So according to our convention in Appendix A, $W_s(\Lambda, \eta)$ must furnish a unitary representation of $SO(D-1)$.

The general Wigner rotation $W(\Lambda, \eta)$ can be worked out using the following equation:

$$W_s(\Lambda, \eta)\gamma^\mu W_s^{-1}(\Lambda, \eta) = W_\nu{}^\mu(\Lambda, \eta)\gamma^\nu \quad (2.22)$$

It is easy to verify that

$$W_s(\Lambda, \eta)\gamma^0 W_s^{-1}(\Lambda, \eta) = \gamma^0, \quad (2.23)$$

that is,

$$W_0{}^0(\Lambda, \eta) = 1, \quad W_i{}^0(\Lambda, \eta) = 0. \quad (2.24)$$

Using (2.2), (2.7), and the commutation relations in Appendix A, we obtain

$$\begin{aligned} W_s(\Lambda, \eta)\gamma^i W_s^{-1}(\Lambda, \eta) &= W_\nu{}^i(\Lambda, \eta)\gamma^\nu = W_j{}^i(\Lambda, \eta)\gamma^j \\ &= \left(-\frac{[\Lambda L(\eta)]_0{}^i [\Lambda L(\eta)]_j{}^0}{1 + [\Lambda L(\eta)]_0{}^0} + [\Lambda L(\eta)]_j{}^i \right) \gamma^j. \end{aligned} \quad (2.25)$$

In summary,

$$\begin{aligned} W_0{}^0(\Lambda, p) &= 1, \\ W_i{}^0(\Lambda, p) &= W_0{}^i(\Lambda, p) = 0, \\ W_j{}^i(\Lambda, p) &= -\frac{[\Lambda L(p)]_0{}^i [\Lambda L(p)]_j{}^0}{1 + [\Lambda L(p)]_0{}^0} + [\Lambda L(p)]_j{}^i. \end{aligned} \quad (2.26)$$

We see that once the explicit expression for Λ is given, one can calculate $W_j{}^i(\Lambda, p)$ immediately, without having to work out the explicit expression of Λ_s .

Using (2.5), a short calculation gives

$$\begin{aligned} W_j{}^i(\Lambda, p) &= \frac{[-\Lambda_0{}^0 p^i/M + \Lambda_0{}^i + (\gamma - 1)\Lambda_0{}^k \hat{p}_k \hat{p}^i](\Lambda p)_j}{M + (\Lambda p)^0} \\ &\quad - \Lambda_j{}^0 p^i/M + (\gamma - 1)\Lambda_j{}^k \hat{p}_k \hat{p}^i + \Lambda_j{}^i. \end{aligned} \quad (2.27)$$

The Wigner rotation (2.26) can be also derived without relying on Clifford algebra. It is not difficult to construct $L(\Lambda p)$:

$$\begin{aligned} L^i{}_j(\Lambda p) &= \delta^{ij} + (\gamma_\Lambda - 1)\widehat{\Lambda p}^i \widehat{\Lambda p}^j, \\ L^0{}_i(\Lambda p) &= L^i{}_0(\Lambda p) = \widehat{\Lambda p}^i \sqrt{\gamma_\Lambda^2 - 1}, \\ L^0{}_0(\Lambda p) &= \gamma_\Lambda, \end{aligned} \quad (2.28)$$

where

$$\begin{aligned} \gamma_\Lambda &= (\Lambda p)^0/M = [\Lambda L(p)]^0{}_0, \\ \widehat{\Lambda p}^i &= \frac{(\Lambda p)^i}{\sqrt{(\Lambda p)^j(\Lambda p)^j}} = \frac{[\Lambda L(p)]^i{}_0}{\sqrt{\gamma_\Lambda^2 - 1}} \end{aligned} \quad (2.29)$$

Using $(L^{-1})^\mu{}_\nu(\Lambda p) = \eta^{\mu\rho}\eta_{\nu\sigma}L^\sigma{}_\rho(\Lambda p)$ and eqs. (2.29), we find that

$$\begin{aligned}(L^{-1})^i{}_j(\Lambda p) &= \delta^{ij} + \frac{[\Lambda L(p)]^i{}_0[\Lambda L(p)]^j{}_0}{[\Lambda L(p)]^0{}_0 + 1}, \\ (L^{-1})^0{}_i(\Lambda p) &= (L^{-1})^i{}_0(\Lambda p) = -[\Lambda L(p)]^i{}_0, \\ (L^{-1})^0{}_0(\Lambda p) &= [\Lambda L(p)]^0{}_0,\end{aligned}\tag{2.30}$$

Substituting (2.30) into the equation

$$W^\mu{}_\nu(\Lambda, p) = (L^{-1})^\mu{}_\rho(\Lambda p)\Lambda^\rho{}_\sigma L^\sigma{}_\nu(p),\tag{2.31}$$

after a slightly length algebra, one obtains

$$\begin{aligned}W^0{}_0(\Lambda, p) &= 1, \\ W^i{}_0(\Lambda, p) &= W^0{}_i(\Lambda, p) = 0, \\ W^j{}_i(\Lambda, p) &= -\frac{[\Lambda L(p)]^0{}_i[\Lambda L(p)]^j{}_0}{1 + [\Lambda L(p)]^0{}_0} + [\Lambda L(p)]^j{}_i,\end{aligned}\tag{2.32}$$

which are exactly the same as (2.26).

Using $\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\eta^{\rho\sigma} = \eta^{\mu\nu}$ and $L^\mu{}_\rho L^\nu{}_\sigma\eta^{\rho\sigma} = \eta^{\mu\nu}$, it is not difficult to verify that

$$W^k{}_i(\Lambda, p)W^i{}_j(\Lambda, p) = \delta_{ij}.\tag{2.33}$$

Namely, the little group is indeed $SO(D-1)$.

We now proceed to discuss two important special cases: Λ is a general pure boost or a general pure rotation.

If Λ_s is a pure rotation, i.e., $\Lambda_s = R_s$, then $(R_s^\dagger)^{-1} = R_s$ (see (A.9) and (A.5)). Plugging it into equation (2.19), and using (2.3), we learn that

$$W_s(R, \eta) = \frac{R_s[L_s^{-1}(\eta) + L_s(\eta)]}{2 \cosh(\eta_\Lambda/2)} = \frac{\cosh(\eta/2)}{\cosh(\eta_\Lambda/2)} R_s = R_s.\tag{2.34}$$

In the last equity, $\cosh(\eta_\Lambda/2) = \cosh(\eta/2)$ can be proved as follows: If $\Lambda = R$, one has $\Lambda_0^0 = 1$ and $\Lambda_0^i = 0$; Plugging them into the first equation of (2.14) proves $\cosh(\eta_\Lambda) = \cosh(\eta)$. Using (2.34), we find that

$$W_s(R, \eta)\gamma^i W_s^{-1}(R, \eta) = W_\mu{}^i(R, \eta)\gamma^\mu = R_s\gamma^i R_s^{-1} = R_j{}^i\gamma^j\tag{2.35}$$

Namely, $W_0{}^i(R, \eta) = 0$ and $W_j{}^i(R, \eta) = R_j{}^i$. That is

$$W(R, \eta) = R.\tag{2.36}$$

(One can also prove the above equation by substituting $\Lambda = R$ into (2.26).) In other words, if Λ is an arbitrary pure rotation R , the Wigner rotation $W(R, \eta)$ is exactly the same as R , independent of the parameter η or momentum p . In 4D, the above important equation is proved by using a different method [2]. We see that in D dimensions, this equation still holds.

However, we have to emphasize that $W(R, \eta) = R$ is due to the particular “standard boost” (2.4) or (2.6). If we use another “standard boost” $\tilde{L}(p)$, satisfying $\tilde{L}(p)^\mu{}_\nu k^\nu = L(p)^\mu{}_\nu k^\nu = p^\mu$, but $\tilde{L}(p) \neq L(p) = (2.6)$, it is possible that $\tilde{W}(R, \eta) \neq R$. This can be seen as follows: According to (1.2),

$$\tilde{W}(R, p) = S(\Lambda p)W(R, p)S^{-1}(p) = S(\Lambda p)RS^{-1}(p) \quad (2.37)$$

where $S(p) \equiv \tilde{L}^{-1}(p)L(p)$; Generally speaking, $S(\Lambda p)RS^{-1}(p) \neq R$.

If Λ_s is a pure boost, i.e., $\Lambda_s = L_s(\xi)$, then $L_s^\dagger(\xi) = L_s(\xi)$ (see (A.9) and (A.5)). Plugging this equation into (2.19), we obtain

$$W_s(\xi, \eta) \equiv W_s(\Lambda, \eta)|_{\Lambda=L(\xi)} = \frac{L_s^{-1}(\xi)L_s^{-1}(\eta) + L_s(\xi)L_s(\eta)}{\sqrt{2(1 + [L_s(\xi)L(\eta)]_0^0)}}. \quad (2.38)$$

Using (2.3) and (2.4), a short calculation gives

$$W_s(\xi, \eta) = \cos\left(\frac{\Theta}{2}\right) + \sin\left(\frac{\Theta}{2}\right) \frac{2\hat{\xi}_i \hat{\eta}_j \Sigma^{ij}}{\sqrt{1 - (\hat{\xi} \cdot \hat{\eta})^2}} = \exp\left(\Theta \frac{\hat{\xi}_i \hat{\eta}_j \Sigma^{ij}}{\sqrt{1 - (\hat{\xi} \cdot \hat{\eta})^2}}\right), \quad (2.39)$$

where Θ is defined via the equation

$$\tan\left(\frac{\Theta}{2}\right) = \frac{\sinh(\xi/2) \sinh(\eta/2) \sqrt{1 - (\hat{\xi} \cdot \hat{\eta})^2}}{\cosh(\xi/2) \cosh(\eta/2) + (\hat{\xi} \cdot \hat{\eta}) \sinh(\xi/2) \sinh(\eta/2)}. \quad (2.40)$$

Note that $W_s(\xi, \eta)$ is invariant under the discrete transformation $\eta \rightarrow -\xi$ and $\xi \rightarrow \eta$, or $\eta \rightarrow \xi$ and $\xi \rightarrow -\eta$ (see (2.39)):

$$W_s(\xi, \eta) = W_s(\eta, -\xi) = W_s(-\eta, \xi). \quad (2.41)$$

Using

$$W_s(\xi, \eta) \gamma^i W_s^{-1}(\xi, \eta) = W_j^i(\xi, \eta) \gamma^j \quad (2.42)$$

and Eq. (A.6), we find that

$$W_j^i(\xi, \eta) = \exp\left(\Theta \frac{\hat{\xi}_k \hat{\eta}_l \tau^{kl}}{\sqrt{1 - (\hat{\xi} \cdot \hat{\eta})^2}}\right)_j^i, \quad (2.43)$$

where

$$(\tau^{kl})_j^i = \delta^{li} \delta_j^k - \delta^{ki} \delta_j^l \quad (2.44)$$

is the set of $SO(D-1)$ matrices, defined via Eq. (A.6). We see that $W_j^i(\xi, \eta)$ is a rotation on the η - ξ plane, satisfying $W_j^i(\xi, \eta) = W_j^i(\eta, -\xi) = W_j^i(-\eta, \xi)$. The explicit expression

of $W_j^i(\xi, \eta)$ can be worked out by either plugging (2.40) into (2.42), or expanding (2.43) directly:

$$\begin{aligned}
W_j^i(\xi, \eta) &= \delta_j^i + \sin \Theta \frac{\hat{\xi}_k \hat{\eta}_l}{\sqrt{1 - (\hat{\xi} \cdot \hat{\eta})^2}} (\tau^{kl})_j^i + 2(1 - \cos \Theta) \frac{(\hat{\xi}_m \hat{\eta}_n)(\hat{\xi}_k \hat{\eta}_l)}{1 - (\hat{\xi} \cdot \hat{\eta})^2} (\tau^{mn} \tau^{kl})_j^i \\
&= \delta_j^i + \frac{(\cosh \eta - 1)(\cosh \xi - 1)[2(\hat{\eta} \cdot \hat{\xi})\hat{\eta}^{(i}\hat{\xi}^{j)} - (\hat{\xi}^i \hat{\xi}^j + \hat{\eta}^i \hat{\eta}^j)]}{1 + \cosh \eta \cosh \xi + (\hat{\eta} \cdot \hat{\xi}) \sinh \eta \sinh \xi} \\
&\quad - \frac{2\hat{\eta}^{[i}\hat{\xi}^{j]}[\sinh \eta \sinh \xi + (\cosh \eta - 1)(\cosh \xi - 1)(\hat{\eta} \cdot \hat{\xi})]}{1 + \cosh \eta \cosh \xi + (\hat{\eta} \cdot \hat{\xi}) \sinh \eta \sinh \xi}, \tag{2.45}
\end{aligned}$$

where $\hat{\eta}^{(i}\hat{\xi}^{j)} = (\hat{\eta}^i \hat{\xi}^j + \hat{\eta}^j \hat{\xi}^i)/2$ and $\hat{\eta}^{[i}\hat{\xi}^{j]} = (\hat{\eta}^i \hat{\xi}^j - \hat{\eta}^j \hat{\xi}^i)/2$. In deriving (2.45), we have used (2.40). The Wigner rotation (2.45) can be also worked out by substituting the pure boost $\Lambda = L(\xi)$ into the general Wigner rotation (2.26).

2.2 4 Dimensions

It is useful to define $\gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3$, that is $\gamma^{[\mu} \gamma^\nu \gamma^\rho \gamma^{\sigma]} = -\varepsilon^{\mu\nu\rho\sigma} \gamma_5$, where $\varepsilon^{0123} = -\varepsilon_{0123} = 1$. With these definitions, we have

$$\Sigma^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\rho\sigma} \gamma_5. \tag{2.46}$$

To simplify calculations, we decompose the generators $\Sigma^{\mu\nu}$ and the parameters $\omega^{\mu\nu}$ as follows:

$$\Sigma_{\pm}^{\mu\nu} = \frac{1}{2} \left(\Sigma^{\mu\nu} \pm \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\rho\sigma} \right) \tag{2.47}$$

$$\omega_{\pm}^{\mu\nu} = \frac{1}{2} \left(\omega^{\mu\nu} \pm \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \omega_{\rho\sigma} \right) \tag{2.48}$$

Notice that they satisfy the duality conditions

$$\Sigma_{\pm}^{\mu\nu} = \pm \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\pm\rho\sigma} = \frac{1}{2} (1 \mp i\gamma_5) \Sigma^{\mu\nu} \tag{2.49}$$

$$\omega_{\pm}^{\mu\nu} = \pm \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \omega_{\pm\rho\sigma} \tag{2.50}$$

or

$$\Sigma_{\pm ij} = \mp i \varepsilon^{ijk} \Sigma_{\pm k0}, \quad \omega_{\pm ij} = \mp i \varepsilon^{ijk} \omega_{\pm k0}. \tag{2.51}$$

Define

$$\omega_{\pm} = \sqrt{\omega_{\pm\mu\nu} \omega_{\pm}^{\mu\nu}} \quad \text{and} \quad \hat{\omega}_{\pm\mu\nu} = \frac{\omega_{\pm\mu\nu}}{\omega_{\pm}}. \tag{2.52}$$

A calculation gives

$$\Lambda_{\pm s} \equiv \exp \left(\frac{1}{2} \omega_{\pm\mu\nu} \Sigma_{\pm}^{\mu\nu} \right) = \frac{1}{2} (1 \mp i\gamma_5) \cos \frac{\omega_{\pm}}{2} + \hat{\omega}_{\pm\mu\nu} \Sigma_{\pm}^{\mu\nu} \sin \frac{\omega_{\pm}}{2} + \frac{1}{2} (1 \pm i\gamma_5). \tag{2.53}$$

Using $\Lambda_{\pm s}$, it is not difficult to work out the general lorentz transformation,

$$\begin{aligned}\Lambda_s &= \exp\left(\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) = \Lambda_{+s}\Lambda_{-s} \\ &= \frac{1}{2}(1 - i\gamma_5) \cos\frac{\omega_+}{2} + \frac{1}{2}(1 + i\gamma_5) \cos\frac{\omega_-}{2} \\ &\quad + \hat{\omega}_{+\mu\nu}\Sigma_+^{\mu\nu} \sin\frac{\omega_+}{2} + \hat{\omega}_{-\mu\nu}\Sigma_-^{\mu\nu} \sin\frac{\omega_-}{2}.\end{aligned}\quad (2.54)$$

Using

$$\Lambda_{\pm s}\gamma^\nu\Lambda_{\pm s}^{-1} = \Lambda_{\pm\mu}{}^\nu\gamma^\mu, \quad (2.55)$$

one obtains

$$\Lambda_{\pm\mu}{}^\nu = \cos\left(\frac{\omega_\pm}{2}\right)\delta_\mu{}^\nu + 2\sin\left(\frac{\omega_\pm}{2}\right)\hat{\omega}_{\pm\mu}{}^\nu. \quad (2.56)$$

That is

$$\begin{aligned}\Lambda_{\pm 0}{}^0 &= \cos(\omega_\pm/2) \\ \Lambda_{\pm 0}{}^i &= \Lambda_{\pm i}{}^0 = -2\sin(\omega_\pm/2)\hat{\omega}_{\pm i 0} \\ \Lambda_{\pm i}{}^j &= \cos(\omega_\pm/2)\delta^{ij} \mp 2i\sin(\omega_\pm/2)\varepsilon^{ijk}\hat{\omega}_{\pm k 0}\end{aligned}\quad (2.57)$$

If it is a pure boost, i.e., $\omega_{ij} = 0$ and $\omega_{i0} = \eta^i \neq 0$, then

$$\begin{aligned}L_{\pm 0}{}^0(\eta) &= \cosh(\eta/2) \\ L_{\pm 0}{}^i(\eta) &= L_{\pm i}{}^0(\eta) = -\sinh(\eta/2)\hat{\eta}^i \\ L_{\pm i}{}^j(\eta) &= \cosh(\eta/2)\delta^{ij} \mp i\sinh(\eta/2)\varepsilon^{ijk}\hat{\eta}^k\end{aligned}\quad (2.58)$$

Note that

$$\Lambda_s\gamma^\nu\Lambda_s^{-1} = (\Lambda_{+s}\Lambda_{-s})\gamma^\nu(\Lambda_{-s}^{-1}\Lambda_{+s}^{-1}) = (\Lambda_+\Lambda_-)_\mu{}^\nu\gamma^\mu = (\Lambda_-\Lambda_+)_\mu{}^\nu\gamma^\mu. \quad (2.59)$$

So

$$\begin{aligned}\Lambda_\mu{}^\nu &= (\Lambda_-\Lambda_+)_\mu{}^\nu \\ &= \cos\left(\frac{\omega_+}{2}\right)\cos\left(\frac{\omega_-}{2}\right)\delta_\mu{}^\nu + 2\cos\left(\frac{\omega_+}{2}\right)\sin\left(\frac{\omega_-}{2}\right)\hat{\omega}_{-\mu}{}^\nu \\ &\quad + 2\cos\left(\frac{\omega_-}{2}\right)\sin\left(\frac{\omega_+}{2}\right)\hat{\omega}_{+\mu}{}^\nu + 4\sin\left(\frac{\omega_+}{2}\right)\sin\left(\frac{\omega_-}{2}\right)(\hat{\omega}_+\hat{\omega}_-)_\mu{}^\nu\end{aligned}\quad (2.60)$$

The standard Lorentz transformation (2.4) can be also derived using $L_\mu{}^\nu(\eta) = (L_-L_+)_\mu{}^\nu(\eta)$ and (2.58).

We now would like to work out the explicit expression of the spinor representation of the little group (2.19). We expect that Eq. (2.19) takes the “standard” form

$$W_s(\Lambda, \eta) = \frac{\gamma^0\Lambda_s L_s(\eta)(\gamma^0)^{-1} + \Lambda_s L_s(\eta)}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}} = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\hat{\Theta}_i(2\Sigma_i) \quad (2.61)$$

where

$$\Sigma_i \equiv \frac{1}{2}\varepsilon_{ijk}\Sigma^{jk}, \quad \Theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\Theta^{jk}, \quad \hat{\Theta}_i \equiv \Theta_i/\sqrt{\Theta^2}, \quad (2.62)$$

and $\Theta^i = \Theta^i(\Lambda, \eta)$ is a function of $\Lambda_\mu{}^\nu$ and η^i . Let's calculate $\Lambda_s L_s(\eta)$ first. According to (2.56), it must take the general form

$$\begin{aligned} \Lambda_s L_s(\eta) = & \frac{1}{2}(1 - i\gamma_5) \cos \frac{\alpha_+}{2} + \frac{1}{2}(1 + i\gamma_5) \cos \frac{\alpha_-}{2} \\ & + \hat{\alpha}_{+\mu\nu} \Sigma_+^{\mu\nu} \sin \frac{\alpha_+}{2} + \hat{\alpha}_{-\mu\nu} \Sigma_-^{\mu\nu} \sin \frac{\alpha_-}{2}. \end{aligned} \quad (2.63)$$

where the new parameters $\hat{\alpha}_{\pm\mu\nu} = \hat{\alpha}_{\pm\mu\nu}(\omega, \eta)$ and $\alpha_\pm = \alpha_\pm(\omega, \eta)$ are functions of $\omega_{\mu\nu}$ and η_i , whose definitions and properties are similar to that of $\hat{\omega}_{\pm\mu\nu}$ and ω_\pm (see (2.48), (2.50), and (2.52)). Using (2.63), a straightforward calculation gives

$$W_s(\Lambda, \eta) = \frac{(\cos \frac{\alpha_+}{2} + \cos \frac{\alpha_-}{2}) - 2i(\sin \frac{\alpha_+}{2} \hat{\alpha}_{+i0} + \sin \frac{\alpha_-}{2} \hat{\alpha}_{-i0})(2\Sigma_i)}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}} \quad (2.64)$$

We now must determine the relations of $\alpha_{\pm\mu\nu}$ between $\omega_{\pm\mu\nu}$ and η_i . According to (2.56), the vector representations of $\Lambda_\pm L_\pm(\eta)$ are given by

$$(\Lambda_\pm L_\pm(\eta))_\mu{}^\nu = \cos \left(\frac{\alpha_\pm}{2} \right) \delta_\mu{}^\nu + 2 \sin \left(\frac{\alpha_\pm}{2} \right) \hat{\alpha}_{\pm\mu}{}^\nu. \quad (2.65)$$

Substituting (2.57) and (2.58) into (2.65), one obtains

$$(\Lambda_\pm L_\pm(\eta))_0^0 = \cos \frac{\alpha_\pm}{2} = \cos \frac{\omega_\pm}{2} \cosh \frac{\eta}{2} + 2 \sin \frac{\omega_\pm}{2} \sinh \frac{\eta}{2} (\hat{\omega}_\pm \cdot \hat{\eta}), \quad (2.66)$$

$$\begin{aligned} & (\Lambda_\pm L_\pm(\eta))_0^i \\ &= -2 \sin \frac{\alpha_\pm}{2} \hat{\alpha}_{\pm i0} = -\cos \frac{\omega_\pm}{2} \sinh \frac{\eta}{2} \eta_i - 2 \sin \frac{\omega_\pm}{2} \cosh \frac{\eta}{2} \hat{\omega}_{\pm i0} \mp 2i \sin \frac{\omega_\pm}{2} \sinh \frac{\eta}{2} (\hat{\omega}_\pm \times \hat{\eta})_i \end{aligned} \quad (2.67)$$

where $\hat{\omega}_\pm \cdot \hat{\eta} \equiv \hat{\omega}_{\pm i0} \hat{\eta}_i$ and $(\hat{\omega}_\pm \times \hat{\eta})_i = \varepsilon_{ijk} \omega_{\pm j0} \hat{\eta}_k$. Using the above two equations, all terms in the numerator of (2.64) can be also expressed in terms of $\omega_{\pm\mu\nu}$ and η_i .

Using (2.66) and (2.67), we see that (2.64) also takes the following form:

$$W_s(\Lambda, \eta) = \frac{(\Lambda_+ L_+(\eta))_0^0 + (\Lambda_- L_-(\eta))_0^0}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}} + i \frac{(\Lambda_+ L_+(\eta))_0^i + (\Lambda_- L_-(\eta))_0^i}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}} (2\Sigma_i). \quad (2.68)$$

Here

$$\sqrt{2(1 + (\Lambda L)_0^0)} = \sqrt{2[1 + \Lambda_0^0 \cosh(\eta) - \Lambda_0^i \hat{\eta}_i \sinh(\eta)]} \quad (2.69)$$

(See the first equation of (2.14)).

Using (2.66), (2.67), and (2.69), Eq. (2.61) or (2.68) can be calculated straightforwardly:

$$W_s(\Lambda, \eta) = \cos \frac{\Theta}{2} + \sin \frac{\Theta}{2} \hat{\Theta}_i (2\Sigma_i) = \exp(\Theta \hat{\Theta}_i \Sigma_i), \quad (2.70)$$

where

$$\begin{aligned}\cos \frac{\Theta}{2} &= \frac{(\Lambda_+ L_+(\eta))_0^0 + (\Lambda_- L_-(\eta))_0^0}{\sqrt{2(1 + [\Lambda L(\eta)]_0^0)}} \\ &= \frac{[(\cos \frac{\omega_+}{2} + \cos \frac{\omega_-}{2}) \cosh \frac{\eta}{2} + 2(\sin \frac{\omega_+}{2}(\hat{\omega}_+ \cdot \hat{\eta}) + \sin \frac{\omega_-}{2}(\hat{\omega}_- \cdot \hat{\eta})) \sinh \frac{\eta}{2}]}{\sqrt{2[1 + \Lambda_0^0 \cosh(\eta) - \Lambda_0^i \hat{\eta}_i \sinh(\eta)]}},\end{aligned}\quad (2.71)$$

and

$$\begin{aligned}\sin \frac{\Theta}{2} \hat{\Theta}_i &= i \frac{(\Lambda_+ L_+(\eta))_0^i + (\Lambda_- L_-(\eta))_0^i}{\sqrt{2(1 + [\Lambda L(\eta)]_0^0)}} \\ &= \frac{1}{\sqrt{2[1 + \Lambda_0^0 \cosh(\eta) - \Lambda_0^i \hat{\eta}_i \sinh(\eta)]}} \left[-i \left(\left(\cos \frac{\omega_+}{2} - \cos \frac{\omega_-}{2} \right) \sinh \frac{\eta}{2} \hat{\eta}_i \right. \right. \\ &\quad \left. \left. + 2(\sin \frac{\omega_+}{2} \hat{\omega}_{+i0} + \sin \frac{\omega_-}{2} \hat{\omega}_{-i0}) \cosh \frac{\eta}{2} + 2i(\sin \frac{\omega_+}{2} \hat{\omega}_{+j0} + \sin \frac{\omega_-}{2} \hat{\omega}_{-j0}) \varepsilon_{ijk} \eta_k \sinh \frac{\eta}{2} \right) \right].\end{aligned}\quad (2.72)$$

In (2.71) and (2.72), the set of parameters η^i is related to the momentum \vec{p} and mass M via (2.5), and the relation of the general Lorentz transformation $\Lambda_\mu{}^\nu$ with the set of parameters $\omega_{\mu\nu}$ is given by (2.60).

Note that the above equation provide a third way to construct the vector representation of the little group (2.26) in 4 dimensional spacetime. Since now $\cos \frac{\Theta}{2}$ and $\sin \frac{\Theta}{2} \hat{\Theta}_i$ have been worked out completely, it is not difficult to calculate

$$W_j^i(\Lambda, \eta) = \left(\exp(\Theta \hat{\Theta}_k \tau^k) \right)_j^i = \cos \Theta \delta_{ji} + (1 - \cos \Theta) \hat{\Theta}_j \hat{\Theta}_i + \sin \Theta \varepsilon_{jik} \Theta^k, \quad (2.73)$$

where $\tau^k = \frac{1}{2} \varepsilon^{kij} \tau_{ij}$, with $(\tau_{ij})_{kl} = \delta_{ik} \delta_{jl} - \delta_{jk} \delta_{il}$. Plugging the data of (2.71) and (2.72) into (2.73), a slightly length calculation gives

$$W_j^i(\Lambda, \eta) = -\frac{[\Lambda L(\eta)]_0^i [\Lambda L(\eta)]_j^0}{1 + [\Lambda L(\eta)]_0^0} + [\Lambda L(\eta)]_j^i, \quad (2.74)$$

which is exactly the same as (2.26) or (2.27), with $i, j = 1, 2, 3$.

The Wigner rotation in any irreducible representation can be constructed by replacing $\tau_i \rightarrow -iJ_i$ in the right-hand side of the first equity of (2.73):

$$W_{m'm}^{(j)}(\Lambda, \eta) \equiv W_{m'm}^{(j)}\left(\Theta(\Lambda, \eta)\right) = \left(\exp(-i\Theta \hat{\Theta}_k J_k^{(j)}) \right)_{m'm}. \quad (2.75)$$

Here the (irreducible) unitary representations of J_i are given by

$$(J_3^{(j)})_{m'm} = m\hbar \delta_{m'm}, \quad (J_1^{(j)} \pm iJ_2^{(j)})_{m'm} = \hbar \delta_{m', m \pm 1} \sqrt{(j \pm m + 1)(j \mp m)}, \quad (2.76)$$

where $m', m = j, j-1, \dots, -(j-1), -j$. The Wigner's formula for d-function may be useful in calculating $W_{m'm}^{(j)}(\Lambda, \eta)$. For instance, in the special case of $\hat{\Theta}_k = \hat{y}$ or $\Theta \hat{\Theta}_k J_k^{(j)} = \Theta J_2^{(j)}$,

Eq. (2.75) is nothing but the the Wigner's d-function [3]:

$$W_{m'm}^{(j)}\left(\Theta(\Lambda, \eta)\right) = \sum_k (-1)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(j-m+m')!} \\ \times \left(\cos \frac{\Theta}{2}\right)^{2j-2k+m-m'} \left(\sin \frac{\Theta}{2}\right)^{2k-m+m'}, \quad (2.77)$$

where the expressions of $\cos \frac{\Theta}{2}$ and $\sin \frac{\Theta}{2}$ are given by (2.71) and (2.72).

2.3 Summary of This Section

In summary, in D dimensions, the spinor representation of the Wigner rotation is given by

$$W_s(\Lambda, \eta) = \frac{\gamma^0 \Lambda_s L_s(\eta) (\gamma^0)^{-1} + \Lambda_s L_s(\eta)}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}}, \quad (2.78)$$

and the vector representation of the Wigner rotation is given by

$$W_j^i(\Lambda, p) = -\frac{[\Lambda L(p)]_0^i [\Lambda L(p)]_j^0}{1 + [\Lambda L(p)]_0^0} + [\Lambda L(p)]_j^i \\ = \frac{[-\Lambda_0^0 p^i / M + \Lambda_0^i + (\gamma - 1) \Lambda_0^k \hat{p}_k \hat{p}^i](\Lambda p)_j}{M + (\Lambda p)^0} \\ - \Lambda_j^0 p^i / M + (\gamma - 1) \Lambda_j^k \hat{p}_k \hat{p}^i + \Lambda_j^i. \quad (2.79)$$

Here $\Lambda_\mu{}^\nu$ is an arbitrary Lorentz transformation, and $L(p)$ or $L(\eta)$ carries the standard D -momentum $k^\mu = (0, 0, \dots, 0, M)$ to p^μ , i.e. $L^\mu{}_\nu(\eta) k^\nu = p^\mu$, with p^μ the D -momentum of the particle of mass M . The explicit expressions of $L(p)$ and $L(\eta)$ are given by (2.4)–(2.6). And Λ_s and $L_s(\eta)$ are spinor counterparts of $\Lambda_\mu{}^\nu$ and $L_\mu{}^\nu(\eta)$, respectively. The explicit expression for $L_s(\eta)$ is given by (2.3).

3 Wigner Rotations for Massless Particles

3.1 D Dimensions

We now turn to the case of massless particles in D -dimensions. We define the standard D -vector of unit energy as

$$k^\mu = (0, 0, \dots, \kappa, \kappa). \quad (3.1)$$

We see that $k_\mu \gamma^\mu = \kappa(-\gamma^0 + \gamma^{D-1})$. It is therefore more convenient to work in the light-cone coordinates (our conventions are summarized in Appendix A):

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\pm \gamma^0 + \gamma^{D-1}), \quad k^\pm = \frac{1}{\sqrt{2}}(k^0 + k^{D-1}). \quad (3.2)$$

In the light-cone coordinates, we have

$$k_\mu \gamma^\mu = k_- \gamma^- = \sqrt{2} \kappa \gamma^-. \quad (3.3)$$

The little group W preserves k^μ , in the sense that $W^\mu{}_\nu k^\nu = k^\mu$. In the spinor space, this is equivalent to require that

$$W_s \gamma^- W_s^{-1} = \gamma^-, \quad (3.4)$$

where W_s is spinor representation of the little group.

We define the “standard Lorentz transformation” in spinor space as follows

$$\begin{aligned} L_s(\lambda) &\equiv \exp(\lambda_a \Sigma^{+a}) \exp(\lambda_- \Sigma^{+-}), \\ &= \cosh \frac{\lambda_-}{2} + e^{-\lambda_-/2} \lambda_a \Sigma^{+a} + 2 \sinh \frac{\lambda_-}{2} \Sigma^{+-}. \end{aligned} \quad (3.5)$$

where the set of generators is $(\Sigma^{+a}, \Sigma^{+-})$, $a = 1, \dots, D-2$, with $\Sigma^{+a} = \frac{1}{\sqrt{2}}(\Sigma^{0a} + \Sigma^{D-1,a})$ and $\Sigma^{+-} = \Sigma^{0,D-1}$, and the parameters are defined as ⁴

$$(\lambda_a, \lambda_-) = (-p_a/p_-, -\ln(p_-/k_-)), \quad (3.6)$$

where $p_- = p^+ \equiv (p^0 + p^{D-1})/\sqrt{2}$. The vector counterpart of (3.5) $L_\mu{}^\nu(\lambda)$, defined via the equation

$$L_s(\lambda) \gamma^\mu L_s^{-1}(\lambda) = L_\nu{}^\mu(\lambda) \gamma^\nu, \quad (3.7)$$

is therefore given by

$$L(\lambda) = \exp(\lambda_a \tau^{+a}) \exp(\lambda_- \tau^{+-}). \quad (3.8)$$

Here $\tau^{+a} = \frac{1}{\sqrt{2}}(\tau^{0a} + \tau^{D-1,a})$ and $\tau^{+-} = \tau^{0,D-1}$. The matrix elements of $\tau^{\mu\nu}$ are defined as $(\tau^{\mu\nu})_\sigma{}^\rho = \delta_\sigma^\mu \eta^{\nu\rho} - \delta_\sigma^\nu \eta^{\mu\rho}$ (see (A.8)). The matrix elements of $L(\lambda)$ can be either read off from (3.7) or calculated directly using (3.8): In the lightcone coordinate system, they are given by

$$\begin{aligned} L_a{}^b(\lambda) &= \delta_a^b, \quad L_a{}^-(\lambda) = \frac{p_a}{k_-}, \quad L_a{}^+(\lambda) = 0, \\ L_-{}^b(\lambda) &= 0, \quad L_-{}^-(\lambda) = \frac{p_-}{k_-}, \quad L_-{}^+(\lambda) = 0, \\ L_+{}^b(\lambda) &= -\frac{p^b}{p_-}, \quad L_+{}^-(\lambda) = \frac{p_+}{k_-}, \quad L_+{}^+(\lambda) = \frac{k_-}{p_-}. \end{aligned} \quad (3.9)$$

It is straightforward to verify that $L(\lambda)$ does bring k^μ to p^μ .

To satisfy (3.4), one defines the Wigner rotation in the spinor space as

$$W_s(\Lambda, \lambda) = L_s^{-1}(\lambda_\Lambda) \Lambda_s L_s(\lambda). \quad (3.10)$$

Here Λ_s is the general Lorentz transformation in the spinor space, and

$$\begin{aligned} L_s^{-1}(\lambda_\Lambda) &= \exp(-\lambda_\Lambda \Sigma^{+-}) \exp(-\lambda_{\Lambda a} \Sigma^{+a}) \\ &= \cosh \frac{\lambda_{\Lambda-}}{2} - e^{-\lambda_{\Lambda-}/2} \lambda_{\Lambda a} \Sigma^{+a} - 2 \sinh \frac{\lambda_{\Lambda-}}{2} \Sigma^{+-}. \end{aligned} \quad (3.11)$$

⁴For a massless particle of unit energy, $k_- = \sqrt{2}\kappa = \sqrt{2}$.

where the set of parameters λ_Λ is defined such that $L(\lambda_\Lambda)$ transforms k^μ into $(\Lambda p)^\mu$, i.e.,

$$(\lambda_{\Lambda a}, \lambda_{\Lambda -}) = \left(-\frac{(\Lambda p)_a}{(\Lambda p)_-}, -\ln \frac{(\Lambda p)_-}{k_-} \right). \quad (3.12)$$

(The matrix elements of $L(\lambda_\Lambda)$ are given by (3.22).)

The general Wigner rotation $W_\nu^\mu(\Lambda, \lambda)$ can be read off from the following equation:

$$W_s(\Lambda, \lambda) \gamma^\mu W_s^{-1}(\Lambda, \lambda) = W_\nu^\mu(\Lambda, \lambda) \gamma^\nu, \quad (3.13)$$

where in the light-cone coordinates $\gamma^\mu = (\gamma^a, \gamma^-, \gamma^+)$.

First of all, it is not difficult to verify that

$$W_s(\Lambda, \lambda) \gamma^- W_s^{-1}(\Lambda, \lambda) = \gamma^-. \quad (3.14)$$

So

$$W_b^-(\Lambda, \lambda) = W_+^-(\Lambda, \lambda) = 0 \quad \text{and} \quad W_-^-(\Lambda, \lambda) = 1. \quad (3.15)$$

Secondly, after a length calculation, one obtains

$$\begin{aligned} & W_s(\Lambda, \lambda) \gamma^a W_s^{-1}(\Lambda, \lambda) \\ &= [(\Lambda_b^a + \lambda^a \Lambda_b^+) + (\Lambda_-^a + \lambda^a \Lambda_-^+) \lambda_{\Lambda b}] \gamma^b + e^{\lambda_\Lambda} (\Lambda_-^a + \lambda^a \Lambda_-^+) \gamma^-. \end{aligned} \quad (3.16)$$

It can be seen that

$$\begin{aligned} W_+^a(\Lambda, \lambda) &= 0 \\ W_b^a(\Lambda, \lambda) &= (\Lambda_b^a + \lambda^a \Lambda_b^+) + (\Lambda_-^a + \lambda^a \Lambda_-^+) \lambda_{\Lambda b} \\ &= -\frac{[\Lambda L(\lambda)]_a^- [\Lambda L(\lambda)]_-^b}{[\Lambda L(\lambda)]_-^-} + [\Lambda L(\lambda)]_a^b \\ &= \frac{1}{p_- (\Lambda p)_-} \left((p_- \Lambda_b^a - p^a \Lambda_b^+) (\Lambda p)_- - (p_- \Lambda_-^a - p^a \Lambda_-^+) (\Lambda p)_b \right), \\ W_-^a(\Lambda, \lambda) &= e^{\lambda_\Lambda} (\Lambda_-^a + \lambda^a \Lambda_-^+) = \frac{[\Lambda L(\lambda)]_-^a}{[\Lambda L(\lambda)]_-^-}. \end{aligned} \quad (3.17)$$

In calculating Eqs. (3.17), we have used (3.9) and (3.12). (The relation between the standard Lorentz transformation $L(\lambda)$ and the momentum p^μ is given by (3.9).)

Finally, we consider the following equation

$$W_s(\Lambda, \lambda) \gamma^+ W_s^{-1}(\Lambda, \lambda) = W_\nu^+(\Lambda, \lambda) \gamma^\nu. \quad (3.18)$$

We find that the results are

$$\begin{aligned} W_+^+(\Lambda, \lambda) &= 1, \\ W_-^+(\Lambda, \lambda) &= \frac{[\Lambda L(\lambda)]_-^+}{[\Lambda L(\lambda)]_-^-}, \\ W_b^+(\Lambda, \lambda) &= -\frac{[\Lambda L(\lambda)]_-^+ [\Lambda L(\lambda)]_b^-}{[\Lambda L(\lambda)]_-^-} + [\Lambda L(\lambda)]_b^+, \end{aligned} \quad (3.19)$$

where $L(p)$ is defined by (3.9).

Note that the elements in Eqs. (3.19) are *not* independent quantities, in the sense that they can be expressed in terms of the other elements by using the Lorentz transformation

$$W_\mu{}^\rho W_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu}. \quad (3.20)$$

For instance, using $W_b{}^\rho W_-{}^\sigma \eta_{\rho\sigma} = \eta_{b-} = 0$, we obtain that

$$W_b{}^+(\Lambda, \lambda) = -W_b{}^a(\Lambda, \lambda) W_-{}^a(\Lambda, \lambda) = -\frac{[\Lambda L(\lambda)]_-{}^+ [\Lambda L(\lambda)]_b{}^-}{[\Lambda L(\lambda)]_-{}^-} + [\Lambda L(\lambda)]_b{}^+. \quad (3.21)$$

which is exactly the same as the last equation of (3.19). On the other hand, the elements in (3.15) are either 0 or 1, so the only “non-trivial” elements are $W_+{}^a(\Lambda, \lambda)$ and $W_b{}^a(\Lambda, \lambda)$.

Here is another way to calculate the little group element $W_\mu{}^\nu(\Lambda, \lambda)$. First, one can obtain $L_\mu{}^\nu(\lambda_\Lambda)$ by doing the replacements $p^\mu \rightarrow (\Lambda p)^\mu$ and $\lambda \rightarrow \lambda_\Lambda$ in (3.9):

$$\begin{aligned} L_a{}^b(\lambda_\Lambda) &= \delta_a^b, & L_a{}^-(\lambda_\Lambda) &= \frac{(\Lambda p)_a}{k_-}, & L_a{}^+(\lambda_\Lambda) &= 0, \\ L_-{}^b(\lambda_\Lambda) &= 0, & L_-{}^-(\lambda_\Lambda) &= \frac{(\Lambda p)_-}{k_-}, & L_-{}^+(\lambda_\Lambda) &= 0, \\ L_+{}^b(\lambda_\Lambda) &= -\frac{(\Lambda p)_b}{(\Lambda p)_-}, & L_+{}^-(\lambda_\Lambda) &= \frac{(\Lambda p)_+}{k_-}, & L_+{}^+(\lambda_\Lambda) &= \frac{k_-}{(\Lambda p)_-}. \end{aligned} \quad (3.22)$$

Secondly, using the fundamental conditions $L_\mu{}^\rho(\lambda_\Lambda) L_\mu{}^\sigma(\lambda_\Lambda) \eta_{\rho\sigma} = \eta_{\mu\nu}$, it is not difficult to determine the inverse of $L_\mu{}^\nu(\lambda_\Lambda)$:

$$(L^{-1})_\mu{}^\nu(\lambda_\Lambda) = \eta_{\mu\rho} \eta^{\nu\sigma} L_\sigma{}^\rho(\lambda_\Lambda). \quad (3.23)$$

A short calculation gives

$$\begin{aligned} (L^{-1})_a{}^b(\lambda_\Lambda) &= \delta_a^b, & (L^{-1})_a{}^-(\lambda_\Lambda) &= -\frac{(\Lambda p)_a}{(\Lambda p)_-}, & (L^{-1})_a{}^+(\lambda_\Lambda) &= 0, \\ (L^{-1})_-{}^b(\lambda_\Lambda) &= 0, & (L^{-1})_-{}^-(\lambda_\Lambda) &= \frac{\kappa_-}{(\Lambda p)_-}, & (L^{-1})_-{}^+(\lambda_\Lambda) &= 0, \\ (L^{-1})_+{}^b(\lambda_\Lambda) &= \frac{(\Lambda p)_b}{\kappa_-}, & (L^{-1})_+{}^-(\lambda_\Lambda) &= \frac{(\Lambda p)_+}{\kappa_-}, & (L^{-1})_+{}^+(\lambda_\Lambda) &= \frac{(\Lambda p)_-}{\kappa_-}. \end{aligned} \quad (3.24)$$

Finally, one can calculate all matrix elements $W_\mu{}^\nu(\Lambda, \lambda)$ by substituting (3.9) and (3.24) into the equation

$$W(\Lambda, \lambda) = L^{-1}(\lambda_\Lambda) \Lambda L(\lambda). \quad (3.25)$$

For instance, using (3.24), we find that

$$\begin{aligned} &W_b{}^a(\Lambda, \lambda) \\ &= (L^{-1})_b{}^+(\lambda_\Lambda) [\Lambda L(\lambda)]_+{}^a + (L^{-1})_b{}^-(\lambda_\Lambda) [\Lambda L(\lambda)]_-{}^a + (L^{-1})_b{}^c(\lambda_\Lambda) [\Lambda L(\lambda)]_c{}^a \\ &= 0 - \frac{(\Lambda p)_b}{(\Lambda p)_-} [\Lambda L(\lambda)]_-{}^a + \delta_b^c [\Lambda L(\lambda)]_c{}^a \\ &= -\frac{[\Lambda L(\lambda)]_a{}^- [\Lambda L(\lambda)]_-{}^b}{[\Lambda L(\lambda)]_-{}^-} + [\Lambda L(\lambda)]_a{}^b, \end{aligned} \quad (3.26)$$

which is exactly the same as the second equation of (3.17). In the last line, we have used (3.9).

By a length but direct calculation, one can show that

$$W_a{}^c(\Lambda, \lambda)W_b{}^c(\Lambda, \lambda) = \delta_{ab}. \quad (3.27)$$

So $W_b{}^a(\Lambda, \lambda)$ must be the elements of the $SO(D-2)$ subgroup. Hence the group element $W_b{}^a(\Lambda, \lambda)$ is the most important result of this section.

However, we still need to demonstrate that the little group is $ISO(D-2)$. Using (3.16) and $(\gamma^-)^2 = 0$, one obtains immediately

$$W_s(\Lambda, \lambda)A^aW_s^{-1}(\Lambda, \lambda) = W_b{}^a(\Lambda, \lambda)A^b, \quad (3.28)$$

where $A^a = \Sigma^{-a}$ (see (A.17)). On the other hand,

$$\begin{aligned} & W_s(\Lambda, \lambda)\Sigma^{ab}W_s^{-1}(\Lambda, \lambda) \\ &= W_c{}^a(\Lambda, \lambda)W_d{}^b(\Lambda, \lambda)\Sigma^{cd} + (W_-{}^a(\Lambda, \lambda)W_c{}^b(\Lambda, \lambda) - W_-{}^b(\Lambda, \lambda)W_c{}^a(\Lambda, \lambda))A^c. \end{aligned} \quad (3.29)$$

After defining

$$a^a(\Lambda, \lambda) \equiv W_-{}^b(\Lambda, \lambda)W_a{}^b(\Lambda, \lambda) = -W_a{}^+(\Lambda, \lambda), \quad (3.30)$$

(See (3.21).) Eq. (3.29) can be written as

$$W_s(\Lambda, \lambda)\Sigma^{ab}W_s^{-1}(\Lambda, \lambda) = W_c{}^a(\Lambda, \lambda)W_d{}^b(\Lambda, \lambda)\left(\Sigma^{cd} + a^c(\Lambda, \lambda)A^d - a^d(\Lambda, \lambda)A^c\right). \quad (3.31)$$

Eqs. (3.28) and (3.31) are the standard transformation law of the generators of $ISO(D-2)$, and the group can be parameterized as

$$W_s(\Lambda, \lambda) = \exp\left(a^a(\Lambda, \lambda)A^a\right)\exp\left(\frac{1}{2}\Theta_{cd}(\Lambda, \lambda)\Sigma^{cd}\right) \quad (3.32)$$

Here the set of parameters $\Theta_{cd}(\Lambda, \lambda)$ is defined via the equation

$$\exp\left(\frac{1}{2}\Theta_{cd}(\Lambda, \lambda)\tau^{cd}\right)_a{}^b = W_a{}^b(\Lambda, \lambda), \quad (3.33)$$

with $(\tau^{cd})_a{}^b = \delta_a^c\delta^{db} - \delta_a^d\delta^{cb}$.

It is interesting to note that in our construction, the spinor representation of the translation operators A^a satisfy

$$(A^a)^2 = 0, \quad (\text{no sum}) \quad (3.34)$$

where we have used (A.17). So the eigenvalues of A^a are *zero* automatically, without considering the topology of the Lorentz group [2].

Eq. (3.32) suggests that the general representation of the little group takes the form

$$W_{(R)}(\Lambda, \lambda) = \exp\left(a^a(\Lambda, \lambda)T_{(R)}^a\right)\exp\left(\frac{1}{2}\Theta_{cd}(\Lambda, \lambda)J_{(R)}^{cd}\right) \quad (3.35)$$

with $T_{(R)}^a$ and $J_{(R)}^{cd}$ furnishing a representation R of the generators of the $ISO(D-2)$ group. However, to avoid continuous degree of freedom of massless particles, we require that the physical states are eigenstates of $T_{(R)}^a$, but all eigenvalues are zero [2].

3.2 4 Dimensions, and Applications to Gauge Theory

In $4D$, it is relatively easier to determine the angle of the Wigner Rotation $\Theta(\Lambda, \lambda)$:

$$\begin{aligned}\sin(\Theta(\Lambda, \lambda)) &= W_1^2(\Lambda, \lambda) = -W_2^1(\Lambda, \lambda), \\ \cos(\Theta(\Lambda, \lambda)) &= W_1^1(\Lambda, \lambda) = W_2^2(\Lambda, \lambda),\end{aligned}\tag{3.36}$$

where the matrix elements $W_b^a(\Lambda, \lambda)$ ($a, b = 1, 2$) are given by the second equation of (3.17). According to Eq. (3.30), the set of parameters of the translation part of $ISO(2)$ is

$$a^a(\Lambda, p) = -W_a^+ (\Lambda, \lambda),\tag{3.37}$$

whose values can be read off from (3.21) and (3.22).

It is interesting to consider a different “standard Lorentz transformation”. For instance, let us try

$$\tilde{L}(p) = \exp(-\phi\tau^{12})\exp(-\theta\tau^{13})\exp(\lambda\tau^{03}),\tag{3.38}$$

with the parameters relating to the momentum \vec{p} as follows

$$\begin{aligned}\hat{p}^i &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ |\vec{p}| &= \kappa e^{-\lambda}.\end{aligned}\tag{3.39}$$

This $\tilde{L}(p)$ is adopted from the textbook [2], but rewritten using our notation. It can be seen that $\tilde{L}(p)^\mu{}_\nu k^\nu = L(p)^\mu{}_\nu k^\nu = p^\mu$ but $\tilde{L}(p) \neq L(p)$. (Our $L(p)$ is defined by (3.8) and (3.6).) Let

$$\widetilde{W}(\Lambda, p) = \tilde{L}^{-1}(\Lambda p)\Lambda\tilde{L}(p).\tag{3.40}$$

According to Eq. (1.2), we must have

$$\widetilde{W}(\Lambda, p) = S(\Lambda p)W(\Lambda, p)S^{-1}(p).\tag{3.41}$$

Note that

$$S(p) = \tilde{L}^{-1}(p)L(p)\tag{3.42}$$

is itself a little-group, since

$$S^\mu{}_\nu(p)k^\nu = (\tilde{L}^{-1})^\mu{}_\rho(p)L^\rho{}_\nu(p)k^\nu = (\tilde{L}^{-1})^\mu{}_\rho(p)p^\rho = k^\mu.\tag{3.43}$$

In light-cone coordinates, we can decompose Eq. (3.41) into the following two essential parts

$$\widetilde{W}_b^a(\Lambda, p) = S_b^c(\Lambda p)W_c^d(\Lambda, p)(S^{-1})_d^a(p),\tag{3.44}$$

$$\tilde{a}^a(\Lambda, p) = S_a^b(\Lambda p)a^b(\Lambda, p) - S_a^+(\Lambda p) - S_a^b(\Lambda p)W_b^c(\Lambda, p)(S^{-1})_c^+(p).\tag{3.45}$$

In deriving (3.45), we have used the definition $\tilde{a}^a(\Lambda, p) = -\widetilde{W}_a^+(\Lambda, p)$. Eqs (3.44) and (3.45) also hold in D -dimensions.

We now would like to work out $\widetilde{W}_b^a(\Lambda, p)$ ($a, b = 1, 2$). Using (3.38) and (3.39), a direct calculation gives $\widetilde{L}^\mu{}_\nu(p)$: (We set $\kappa = 1$.)

$$\begin{aligned}\widetilde{L}^i{}_0(p) &= \frac{p_0^2 - 1}{2p_0^2} p^i, & \widetilde{L}^0{}_0(p) &= \frac{p_0^2 + 1}{2p_0^2}, & \widetilde{L}^i{}_3(p) &= \frac{p_0^2 + 1}{2p_0^2} p^i, \\ \widetilde{L}^0{}_3(p) &= \frac{p_0^2 - 1}{2p_0^2}, & \widetilde{L}^a{}_1(p) &= \frac{p_3 p^a}{p_0 \sqrt{p_0^2 - p_3^2}}, & \widetilde{L}^a{}_2(p) &= \frac{-\varepsilon_{ab} p^b}{\sqrt{p_0^2 - p_3^2}}, \\ \widetilde{L}^3{}_1(p) &= -\sqrt{1 - \frac{p_3^2}{p_0^2}}, & \widetilde{L}^3{}_2(p) &= \widetilde{L}^0{}_2(p) = \widetilde{L}^0{}_1(p) = 0,\end{aligned}\tag{3.46}$$

where $\varepsilon_{ab} = -\varepsilon_{ba}$ and $\varepsilon_{12} = 1$, and $i = 1, 2, 3$. One can obtain $\widetilde{L}^\mu{}_\nu(\Lambda p)$ from the above equation by simply replacing p^μ by $(\Lambda p)^\mu$. The inverse transformation matrix $(\widetilde{L}^{-1})^\mu{}_\nu(p_\Lambda)$ can be calculated by using the equation $(\widetilde{L}^{-1})^\mu{}_\nu(p_\Lambda) = \eta^{\mu\rho} \eta_{\nu\sigma} \widetilde{L}^\sigma{}_\rho(p_\Lambda)$; Its expression is

$$\begin{aligned}(\widetilde{L}^{-1})^0{}_i(p_\Lambda) &= -\frac{(p_\Lambda^0)^2 - 1}{2(p_\Lambda^0)^2} p_\Lambda^i, & (\widetilde{L}^{-1})^0{}_0(p_\Lambda) &= \frac{(p_\Lambda^0)^2 + 1}{2p_\Lambda^0}, \\ (\widetilde{L}^{-1})^3{}_i(p_\Lambda) &= \frac{(p_\Lambda^0)^2 + 1}{2(p_\Lambda^0)^2} p_\Lambda^i, & (\widetilde{L}^{-1})^3{}_0(p_\Lambda) &= -\frac{(p_\Lambda^0)^2 - 1}{2p_\Lambda^0}, \\ (\widetilde{L}^{-1})^1{}_a(p_\Lambda) &= \frac{p_\Lambda^3 p_\Lambda^a}{p_\Lambda^0 \sqrt{(p_\Lambda^0)^2 - (p_\Lambda^3)^2}}, & (\widetilde{L}^{-1})^2{}_a(p_\Lambda) &= \frac{-\varepsilon_{ab} p_\Lambda^b}{\sqrt{(p_\Lambda^0)^2 - (p_\Lambda^3)^2}}, \\ (\widetilde{L}^{-1})^1{}_3(p_\Lambda) &= -\sqrt{1 - \frac{(p_\Lambda^3)^2}{(p_\Lambda^0)^2}}, & (\widetilde{L}^{-1})^2{}_3(p_\Lambda) &= (\widetilde{L}^{-1})^2{}_0(p_\Lambda) = (\widetilde{L}^{-1})^1{}_0(p_\Lambda) = 0,\end{aligned}\tag{3.47}$$

where p_Λ^μ stands for $(\Lambda p)^\mu$.

In terms of matrix elements, the Wigner rotation (3.40) reads

$$\widetilde{W}^\mu{}_\nu(\Lambda, p) = (\widetilde{L}^{-1})^\mu{}_\rho(p_\Lambda) \Lambda^\rho{}_\sigma \widetilde{L}^\sigma{}_\nu(p).\tag{3.48}$$

Using (3.46) and (3.47), we find that

$$\begin{aligned}\widetilde{W}^1{}_1(\Lambda, p) &\equiv \cos(\widetilde{\Theta}(\Lambda, p)) \\ &= \frac{\widehat{p}_\Lambda^3 \widehat{p}_\Lambda^a [-\Lambda^a{}_3 (1 - \widehat{p}_3^2) + \Lambda^a{}_b \widehat{p}^b \widehat{p}^3] - [1 - (\widehat{p}_\Lambda^3)^2] [-\Lambda^3{}_3 (1 - \widehat{p}_3^2) + \Lambda^3{}_b \widehat{p}^b \widehat{p}^3]}{\sqrt{[1 - (\widehat{p}_\Lambda^3)^2] (1 - \widehat{p}_3^2)}},\end{aligned}\tag{3.49}$$

and

$$\widetilde{W}^1{}_2(\Lambda, p) \equiv \sin(\widetilde{\Theta}(\Lambda, p)) = \frac{\varepsilon_{ab} \widehat{p}^b (\Lambda^3{}_a - \Lambda^0{}_a \widehat{p}_\Lambda^3)}{\sqrt{[1 - (\widehat{p}_\Lambda^3)^2] (1 - \widehat{p}_3^2)}},\tag{3.50}$$

where the unit vector $\widehat{p}^i = p^i/|\vec{p}|$ is the direction of the momentum \vec{p} , and \widehat{p}_Λ^i has a similar definition. We do not have to calculate $\widetilde{W}^2{}_2(\Lambda, p)$ and $\widetilde{W}^2{}_1(\Lambda, p)$, since $\widetilde{W}^2{}_2(\Lambda, p) = \widetilde{W}^1{}_1(\Lambda, p)$ and $\widetilde{W}^2{}_1(\Lambda, p) = -\widetilde{W}^1{}_2(\Lambda, p)$.

Similarly, using (3.46), (3.47), and (3.48), the translation part of $ISO(2)$

$$\widetilde{a}^a(\Lambda, p) = -\widetilde{W}_a{}^+(\Lambda, p)\tag{3.51}$$

(see (3.30)) can be worked out as well.

It is interesting to verify (3.44) and (3.45). One can calculate $S(p) = \tilde{L}^{-1}(p)L(p)$ by using $(\tilde{L}^{-1})^\mu{}_\nu = \eta^{\mu\rho}\eta_{\nu\sigma}\tilde{L}^\sigma{}_\rho$ (with $\tilde{L}^\sigma{}_\rho$ defined by (3.46)) and the definition of $L(p)$ (3.9). And $S^{-1}(\Lambda p) = L^{-1}(\Lambda p)\tilde{L}(\Lambda p)$ can be calculated in a similar way. We have verified (3.44) and (3.45) in the case of infinitesimal Lorentz transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + (\delta\omega)^\mu{}_\nu \quad (3.52)$$

under the condition that $(p^0)^2 - (p^3)^2 \neq 0$.

We now apply our results to the $U(1)$ gauge theory in $4D$. In the interaction picture, the gauge field in $4D$ takes the form [2]

$$a_\mu(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{\sqrt{2p_0}} \sum_{\sigma=\pm 1} \left[e_\mu(\vec{p}, \sigma) e^{ip \cdot x} a(\vec{p}, \sigma) + e_\mu^*(\vec{p}, \sigma) e^{-ip \cdot x} a^\dagger(\vec{p}, \sigma) \right]. \quad (3.53)$$

Here the polarization vector $e^\mu(\vec{p}, \sigma) = L(p)^\mu{}_\nu e^\nu(\vec{k}, \sigma)$, where the standard Lorentz transformation $L(p)^\mu{}_\nu$ is defined by (3.9). Following the convention of [2], we specify the polarization vectors as

$$e^\mu(\vec{k}, \pm 1) = (1, \pm i, 0, 0)/\sqrt{2}$$

where \vec{k} is the standard momentum.

In $4D$, the vector representation of Eq. (3.35) reads

$$W^\mu{}_\nu(\Lambda, p) = \exp(a^a(\Lambda, p)\tau^{-a})^\mu{}_\rho \exp(\Theta(\Lambda, p)\tau^3)^\rho{}_\nu \quad (3.54)$$

where $(\tau^{-a})^\mu{}_\nu = \frac{1}{\sqrt{2}}(-\tau^{0a} + \tau^{3a})^\mu{}_\nu$, $(\tau^3)^\mu{}_\nu = (\tau^{12})^\mu{}_\nu$, and $(\tau^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu$. From now on, the letter a will be reserved for the creation and annihilation operators. Following the convention of [2], we denote the translation parameters of $ISO(2)$ as α and β , namely,

$$a^a(\Lambda, p) = \left(\alpha(\Lambda, p), \beta(\Lambda, p) \right). \quad (3.55)$$

Under an arbitrary Lorentz transformation Λ , the creation and annihilation operators transform as [2]

$$U(\Lambda)a(\vec{p}, \sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{-i\sigma\Theta(\Lambda, p)} a(\vec{p}_\Lambda, \sigma) \quad (3.56)$$

$$U(\Lambda)a^\dagger(\vec{p}, \sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{i\sigma\Theta(\Lambda, p)} a^\dagger(\vec{p}_\Lambda, \sigma) \quad (3.57)$$

Here \vec{p}_Λ stands for $\Lambda^i{}_\mu p^\mu$ or $(\Lambda p)^i$. On the other hand, under the general Lorentz transformation Λ ,

$$\begin{aligned} \Lambda^\mu{}_\nu e^\nu(\vec{p}, \pm 1) &= L^\mu{}_\nu(\Lambda p)(L^{-1}(\Lambda p)\Lambda L(p))^\nu{}_\rho e^\rho(\vec{k}, \pm 1) \\ &= L^\mu{}_\nu(\Lambda p)W^\nu{}_\rho(\Lambda, p)e^\rho(\vec{k}, \pm 1) \\ &= e^{\pm i\Theta(\Lambda, p)} \left(e^\mu(\vec{p}_\Lambda, \pm 1) + \frac{\alpha(\Lambda, p) \pm \beta(\Lambda, p)}{|\vec{k}|} (\Lambda p)^\mu \right) \end{aligned} \quad (3.58)$$

In the last line, we have used (3.54). That is, the polarization vectors cannot transform as a true Lorentz vector [2]:

$$e^{-(\pm i\Theta(\Lambda, p))} e_\mu(\vec{p}, \pm 1) = \Lambda^\nu{}_\mu e_\nu(\vec{p}_\Lambda, \pm 1) + \frac{\alpha(\Lambda, p) \pm i\beta(\Lambda, p)}{|\vec{k}|} p_\mu \quad (3.59)$$

Or, according to Weinberg's notation [2],

$$e^\mu(\vec{p}_\Lambda, \pm 1) e^{\pm i\Theta(\Lambda, p)} = \Lambda^\mu{}_\nu e^\nu(\vec{p}, \pm 1) + (\Lambda p)^\mu \Omega_\pm(\Lambda, p) \quad (3.60)$$

Here $\Omega_\pm(\Lambda, p) \equiv -e^{\pm i\Theta(\Lambda, p)} [\alpha(\Lambda, p) \pm i\beta(\Lambda, p)] / |\vec{k}|$.

So under the Lorentz transformation,

$$U(\Lambda) a_\mu(x) U^{-1}(\Lambda) = \Lambda^\nu{}_\mu a_\nu(\Lambda x) + \partial_\mu \Omega(x, \Lambda) \quad (3.61)$$

where

$$\Omega(x, \Lambda) = -\frac{i}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3 p}{\sqrt{2p_0}} \sum_{\sigma=\pm 1} \left[\frac{\alpha + i\beta}{|\vec{k}|} e^{ip \cdot (\Lambda x)} a(\vec{p}, \sigma) - \frac{\alpha - i\beta}{|\vec{k}|} e^{-ip \cdot (\Lambda x)} a^\dagger(\vec{p}, \sigma) \right] \quad (3.62)$$

If we calculate everything by using

$$\widetilde{W}^\mu{}_\nu(\Lambda, p) = \exp(\widetilde{a}^a(\Lambda, p) \tau^{-a})^\mu{}_\rho \exp(\widetilde{\Theta}(\Lambda, p) \tau^3)^\rho{}_\nu, \quad (3.63)$$

where

$$\widetilde{a}^a(\Lambda, p) = \left(\widetilde{\alpha}(\Lambda, p), \widetilde{\beta}(\Lambda, p) \right), \quad (3.64)$$

in stead of $W(\Lambda, p)$ (see (3.54)), the angle Θ in (3.56) and (3.57) must be replaced by $\widetilde{\Theta}$, and α and β in (3.62) must be replaced by $\widetilde{\alpha}$ and $\widetilde{\beta}$. (One can transform the set of parameters (α, β, Θ) into $(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\Theta})$ by using (3.44) and (3.45).) After making these replacements, *the only change in (3.61) is that $\Omega(x, \Lambda)$ gets replaced by*

$$\widetilde{\Omega}(x, \Lambda) = -\frac{i}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3 p}{\sqrt{2p_0}} \sum_{\sigma=\pm 1} \left[\frac{\widetilde{\alpha} + i\widetilde{\beta}}{|\vec{k}|} e^{ip \cdot (\Lambda x)} a(\vec{p}, \sigma) - \frac{\widetilde{\alpha} - i\widetilde{\beta}}{|\vec{k}|} e^{-ip \cdot (\Lambda x)} a^\dagger(\vec{p}, \sigma) \right] \quad (3.65)$$

namely,

$$U(\Lambda) a_\mu(x) U^{-1}(\Lambda) = \Lambda^\nu{}_\mu a_\nu(\Lambda x) + \partial_\mu \widetilde{\Omega}(x, \Lambda). \quad (3.66)$$

This is the result calculated by using Eq. (3.63), the convention of [2]. We see that (3.61) and (3.66) are only up to a gauge transformation, which is due to the *difference* between two “standard Lorentz transformation”, defined by (3.42). Or in other words, two different “standard Lorentz transformations” can generate a gauge transformation.

3.3 Summary of This Section

In D dimensions, the vector representation of the $SO(D-2)$ part of the Wigner little group $ISO(D-2)$ is given by

$$\begin{aligned} W_b^a(\Lambda, \lambda) &= -\frac{[\Lambda L(\lambda)]_a^- [\Lambda L(\lambda)]_-^b}{[\Lambda L(\lambda)]_-^-} + [\Lambda L(\lambda)]_a^b \\ &= \frac{1}{p_- (\Lambda p)_-} \left((p_- \Lambda_b^a - p^a \Lambda_b^+) (\Lambda p)_- - (p_- \Lambda_-^a - p^a \Lambda_-^+) (\Lambda p)_b \right), \end{aligned} \quad (3.67)$$

and the translation part is defined as

$$\begin{aligned} a^a(\Lambda, p) = -W_a^+(\Lambda, \lambda) &= \frac{[\Lambda L(\lambda)]_-^+ [\Lambda L(\lambda)]_a^-}{[\Lambda L(\lambda)]_-^-} - [\Lambda L(\lambda)]_a^+ \\ &= \sqrt{2}\kappa \left(\frac{\Lambda_-^+ (\Lambda p)^a}{(\Lambda p)_- p_-} - \frac{\Lambda_a^+}{p_-} \right). \end{aligned} \quad (3.68)$$

Here Λ_μ^ν is an arbitrary Lorentz transformation, and the “standard Lorentz transformation” $L(\lambda)$ carries the standard D -momentum $k^\mu = (0, \dots, 0, \kappa, \kappa)$ to p^μ , i.e. $L^\mu_\nu(\lambda) k^\nu = p^\mu$, with p^μ the D -momentum of any massless particle. The matrix $L_\mu^\nu(\lambda)$ is defined by (3.9).

The general form of the little group for massless particles is defined by (3.35) and (3.33).

4 Acknowledgement

This work is supported in part by the National Science Foundation of China (NSFC) under Grant No. 11475016, and supported partially by the Ren-Cai Foundation of Beijing Jiaotong University through Grant No. 2013RC029, and supported partially by the Scientific Research Foundation for Returned Scholars, Ministry of Education of China.

A Conventions and Useful Identities

In this appendix, we introduce our conventions for the gamma matrices and Clifford algebra of $SO(D-1, 1)$. The set of gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \quad (A.1)$$

where $\eta_{00} = -1$ and $\eta_{ij} = \delta_{ij}$. We will use $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$) to raise (lower) indices; For instance, $\gamma^\mu = \eta^{\mu\nu} \gamma_\nu$. The gamma matrices obey the reality conditions

$$\gamma^{0\dagger} = -\gamma^0, \quad \gamma^{i\dagger} = \gamma^i. \quad (A.2)$$

The generators of $SO(D-1, 1)$ are defined as

$$\Sigma^{i0} = \frac{1}{4} [\gamma^i, \gamma^0] \quad (A.3)$$

$$\Sigma^{ij} = \frac{1}{4} [\gamma^i, \gamma^j] \quad (A.4)$$

They obey the reality conditions

$$\begin{aligned}\Sigma^{i0\dagger} &= \gamma^0 \Sigma^{i0} (\gamma^0)^{-1} = \Sigma^{i0}, \\ \Sigma^{ij\dagger} &= \gamma^0 \Sigma^{ij} (\gamma^0)^{-1} = -\Sigma^{ij},\end{aligned}\tag{A.5}$$

and satisfy the commutation relations

$$[\Sigma^{\mu\nu}, \gamma^\rho] = \eta^{\nu\rho} \gamma^\mu - \eta^{\mu\rho} \gamma^\nu \equiv (\tau^{\mu\nu})_\sigma{}^\rho \gamma^\sigma, \tag{A.6}$$

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \eta^{\nu\rho} \Sigma^{\mu\sigma} - \eta^{\mu\rho} \Sigma^{\nu\sigma} - \eta^{\nu\sigma} \Sigma^{\mu\rho} + \eta^{\mu\sigma} \Sigma^{\nu\rho}, \tag{A.7}$$

$$\{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\} = \frac{1}{2}(\gamma^{\mu\nu\rho\sigma} + \eta^{\nu\rho} \eta^{\mu\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}),$$

where $\gamma^{\mu\nu\rho\sigma} \equiv \gamma^{[\mu} \gamma^\nu \gamma^\rho \gamma^{\sigma]} = \frac{1}{4!}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma + \text{permutations})$, and

$$(\tau^{\mu\nu})_\sigma{}^\rho = \delta_\sigma^\mu \eta^{\nu\rho} - \delta_\sigma^\nu \eta^{\mu\rho}. \tag{A.8}$$

In the spinor space, the general rotation and boost are defined as

$$R_s = e^{\frac{1}{2} \omega_{ij} \Sigma^{ij}} \quad \text{and} \quad L_s = e^{\eta_i \Sigma^{i0}}, \tag{A.9}$$

respectively, where ω_{ij} and η_i are parameters, and the subscript “s” stands for spinor representation. Here L_s will be chosen as the standard boost for studying massive (fermionic) particles. A general Lorentz transform will be denoted as Λ_s , obeying the pseudo-reality condition

$$\gamma^0 \Lambda_s^\dagger (\gamma^0)^{-1} = \Lambda_s^{-1}. \tag{A.10}$$

To see this, let us parameterize Λ_s as $\Lambda_s = \exp(\frac{1}{2} \omega_{\mu\nu} \Sigma^{\mu\nu})$, with $\omega_{\mu\nu}$ a set of real antisymmetric tensor. Eq. (A.10) follows by using (A.5).

To describe massless particles, it is more convenient to introduce the light-cone coordinates in D -dimensional spacetime:

$$x^\pm = \frac{1}{\sqrt{2}}(\pm x^0 + x^{D-1}) \tag{A.11}$$

and the transverse space-like coordinates x^a , $a = 1, 2, \dots, D-2$.

In terms of light-cone coordinates, we have

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\pm \gamma^0 + \gamma^{D-1}). \tag{A.12}$$

As a result,

$$\{\gamma^+, \gamma^-\} = 2\eta^{+-} = 2. \tag{A.13}$$

Hence the metric tensor $\eta^{\mu\nu}$ is decomposed into

$$\eta^{+-} = \eta^{-+} = 1, \quad \eta^{ab} = \delta^{ab}, \quad \eta^{++} = \eta^{--} = \eta^{a+} = \eta^{b-} = 0. \tag{A.14}$$

We will use η^{+-} or η_{+-} to raise or lower indices; For instance, $V_- = \eta_{-+}V^+ = V^+$. The inner product of two vectors is defined as

$$V_\mu W^\mu = V^a W_a + V_- V^- + V_+ V^+. \quad (\text{A.15})$$

One can write down the general Lorentz transformation Λ in the light-cone coordinates by using the rules of tensor analysis. For instance,

$$\Lambda_-^+ = \frac{\partial x^\mu}{\partial x^-} \frac{\partial x^+}{\partial x^\nu} \Lambda_\mu^\nu = \frac{1}{2} \left(-\Lambda_0^0 - \Lambda_0^{D-1} + \Lambda_{D-1}^0 + \Lambda_{D-1}^{D-1} \right). \quad (\text{A.16})$$

The generators $\Sigma^{\mu\nu}$ are decomposed into

$$A^a \equiv \Sigma^{-a} = \frac{1}{4} [\gamma^-, \gamma^a], \quad (\text{A.17})$$

$$\Sigma^{+-} = \frac{1}{4} [\gamma^+, \gamma^-] = \Sigma^{0,D-1}, \quad (\text{A.18})$$

$$\Sigma^{+a} = \frac{1}{4} [\gamma^+, \gamma^a], \quad (\text{A.19})$$

$$\Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]. \quad (\text{A.20})$$

Under the above decomposition, the (spinor) algebra of the little group $ISO(D-2)$ reads

$$[A^a, A^b] = 0, \quad (\text{A.21})$$

$$[\Sigma^{ab}, A^c] = \delta^{bc} A^a - \delta^{ac} A^b, \quad (\text{A.22})$$

$$[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc} \Sigma^{ad} - \delta^{ac} \Sigma^{bd} - \delta^{bd} \Sigma^{ac} + \delta^{ad} \Sigma^{bc}. \quad (\text{A.23})$$

Notice that by the definition of A^a (see (A.17)),

$$(A^a)^2 = 0 \quad (\text{A.24})$$

So in the spinor representation, the eigenvalues of A^a are zero automatically.

References

- [1] E. P. Wigner, Ann. Math. **40**, 149 (1939).
- [2] S. Weinberg, “The Quantum Theory of Fields,” Vol. 1: Foundations (ISBN: 978-0-521-67053-1), Cambridge University Press 1995.
- [3] E. P. Wigner, (1931) “Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atomspektren” Braunschweig: Vieweg Verlag. Translated into English by Griffin, J. J. (1959). “Group Theory and its Application to the Quantum Mechanics of Atomic Spectra”. New York: Academic Press.